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EXPONENTIAL LEVELING FOR STOCHASTICALLY PERTURBED DYNAMICAL SYSTEMS

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ABSTRACT

This paper considers solutions of $0 = \varepsilon \sum_{i,j} a^{\varepsilon}_{i,j}(x) u^{\varepsilon}_{x_i x_j} + \sum b^{\varepsilon}_i(x) u^{\varepsilon}_i$ in a bounded domain Ω ~~for~~ which ~~$\sup_{\Omega} |u^{\varepsilon}|$~~ is bounded in $\varepsilon > 0$. We assume that $a^{\varepsilon} \to a^0$, $b^{\varepsilon} \to b^0$ ~~and~~ that all solutions of the ODE ~~$\dot{x} = b^0(x)$,~~ $x(0) \in \Omega$ converge to a single linearly asymptotically stable critical point in Ω without leaving Ω. ~~We give a~~ proof, based on the standard probabilistic interpretation of u^{ε}, of an exponential leveling property, $\sup_{x,y \in K} |u^{\varepsilon}(x) - u^{\varepsilon}(y)| \leq e^{-\delta/\varepsilon}$ for some $\delta > 0$ which depends on the compact set $K \subseteq \Omega$.

EXPONENTIAL LEVELING FOR STOCHASTICALLY PERTURBED DYNAMICAL SYSTEMS

I: Introduction

Consider a deterministic system described by an ordinary differential equation in $\mathbb{R}^d$:

$$dx^0(t) = b(x^0(t))dt. \tag{1.1}$$

A natural model for the behavior of this system, when subjected to a small stochastic perturbation, is the diffusion process described by the Itô equation

$$dx^\varepsilon(t) = b(x^\varepsilon(t))dt + \sqrt{\varepsilon}\ \sigma(x^\varepsilon(t))d\omega_t, \tag{1.2}$$

ω_t being a Brownian motion in $\mathbb{R}^d$. Applications of this type of model can be found in Ludwig [6], Schuss [10] and Matkowsky and Schuss [9]. Several aspects of the asymptotic behavior of $x^\varepsilon(\cdot)$ as $\varepsilon \to 0$ are of interest. Consider in particular a bounded domain Ω. If τ_Ω denotes the exit time of $x^\varepsilon(\cdot)$ from Ω and E_x^ε the expectation for the solution of (1.2) subject to $x^\varepsilon(0) = x$, then (under appropriate regularity assumptions)

$$u^\varepsilon(x) = E_x^\varepsilon[f(x^\varepsilon(\tau_\Omega))]$$

is the solution of the Dirichlet problem

$$0 = \mathscr{L}^{\varepsilon}[u] = \frac{\varepsilon}{2} \sum_{i,j} a_{ij}(x) u_{x_i x_j} + \sum_i b_i(x) u_{x_i} \quad \text{in } \Omega \tag{1.3}$$

$$\text{with } u\big|_{\partial\Omega} = f.$$

(Here $a = \sigma\sigma^T$). The behavior of u^ε as $\varepsilon \downarrow 0$ depends, of course, on the nature of the trajectories of (1.1) which start in Ω. One of the more interesting cases is when all deterministic trajectories starting in Ω remain in Ω and approach a unique stable point, at the origin say. Because all continuous solutions of the reduced equation

$$0 = \sum_i b_i(x) u^0_{x_i}$$

in Ω are constant, one expects that u^ε approaches a constant function, or at least somehow "levels out". We prove here, under modest assumptions, that this leveling does occur and at an exponential rate:

$$\sup_{x,y\in K} |u^\varepsilon(x) - u^\varepsilon(y)| \le e^{-\delta/\varepsilon}$$

for any compact $K \subseteq \Omega$, some $\delta > 0$ and all sufficiently small ε. In many cases much more is known. Matkowsky and Schuss [8] presented a formal calculation to show that u^ε converges to a constant function and derived a formula for what this constant should be. Kamin [5] and Devinatz and Friedman [1] gave rigorous proofs of this in cases where $\mathscr{L}^\varepsilon$ has a self-adjoint form

$$\mathscr{L}^{\varepsilon}[u] = e^{-\psi/\varepsilon} \sum_{i,j} \frac{\partial}{\partial x_i} (e^{\psi/\varepsilon} a_{ij} u_{x_j}). \tag{1.4}$$

In [4] Kamin showed that the formal calculation of Matkowsky and Schuss for (1.3) is correct provided the solutions of certain auxiliary first order PDE's exist and are sufficiently smooth. The fundamental work of Ventcel and Freidlin [12] also establishes that u^{ε} converges to a constant for (1.3) in the case that the variational distance $V(0,y)$ which is central to their treatment attains its minimum over $y \in \partial\Omega$ at a unique place.

Actually the $\mathscr{L}^{\varepsilon}$ in (1.4) is of a more general form than (1.3):

$$\mathscr{L}^{\varepsilon}[u] = \frac{\varepsilon}{2} \sum_{i,j} a_{ij} u_{x_i x_j} + \sum b_i^{\varepsilon} u_{x_i} \tag{1.5}$$

with $b^{\varepsilon} \to b^0$ as $\varepsilon \to 0$. In this context the solutions u^{ε} may not converge to a constant. Indeed, in [1] the authors presented the two examples

$$\varepsilon(x+2)u'' - x(x+2)u' = 0 \tag{1.6a}$$

$$\varepsilon(x+2)u'' + (\varepsilon - x(x+2))u' = 0 \tag{1.6b}$$

on $[-1,1]$, both with $u(-1) = 0$, $u(1) = 1$. They observed that $u^{\varepsilon} \to \frac{1}{2}$ for (1.6a) and $u^{\varepsilon} \to \frac{3}{4}$ for (1.6b). If we combine these two examples as

$$\varepsilon(x+2)u'' + (\varepsilon \operatorname{Sin}(\tfrac{1}{\varepsilon}) - x(x+2))u' = 0, \tag{1.6c}$$

we get an example of the type (1.5) for which u^ε does not converge.

The result proved here is for $\mathscr{L}^\varepsilon$ of the form

$$\mathscr{L}^\varepsilon[2] = \frac{\varepsilon}{2} \sum_{i,j} a^\varepsilon_{i,j} u_{x_i x_j} + \sum b^\varepsilon_i u_{x_i}.$$

The $a^\varepsilon, b^\varepsilon$ are required to converge to a^0, b^0 as $\varepsilon \downarrow 0$. This form of $\mathscr{L}^\varepsilon$ encompasses all the cases (1.3)-(1.6) mentioned. The boundary function $u^\varepsilon\big|_{\partial\Omega} = f^\varepsilon$ is allowed to be ε-dependent and is required only to be bounded (in both x and ε) and measurable (Borel), but need not converge with ε.

Section 2 contains the technical assumptions and the statement of the main theorem. Sections 3 and 4 are devoted to a bound on hitting probabilities which is the cornerstone of our proof. The proof of the theorem is given in Section 4 also. Section 5 contains two additional remarks.

II: Technical Assumptions and Statement of the Main Result

The domain $\Omega \subseteq \mathbb{R}^d$ is assumed to be bounded. To treat u^ε as the solution to an elliptic boundary value problem with $u^\varepsilon|_{\partial\Omega} = f^\varepsilon$, a specified continuous function, one might also want to impose the requirement that $\partial\Omega$ be C^2. The probabilistic definition (2.1) of u^ε renders this unnecessary however. The assumptions on the coefficients are as follows:

a) $a^\varepsilon(x)$, $a^0(x)$ are Lipschitz (or just Hölder) continuous in x uniformly in ε, positive definite symmetric $d \times d$ matrices on $\bar{\Omega}$ and $|a_{ij}^\varepsilon - a_{ij}^0| \to 0$ uniformly on $\bar{\Omega}$ as $\varepsilon \downarrow 0$;

b) $b^\varepsilon(x)$ and $b^0(x)$ are in $C^1(\bar{\Omega})$, $|b^\varepsilon - b^0|$ and $|b_{x_i}^\varepsilon - b_{x_i}^0|$ $(i = 1,\ldots,d)$ all converge to 0 uniformly on $\bar{\Omega}$ as $\varepsilon \downarrow 0$;

c) for any solution to $x^{0\prime}(t) = b^0(x^0(t))$ with $x^0(0) \in \Omega$, $x^0(t) \in \Omega$ for all $t \geq 0$ and $\lim_{t\to+\infty} x^0(t) = 0$;

d) the matrix $B = [\frac{\partial b_i^0(0)}{\partial x_j}]$ is stable, i.e. all its eigenvalues have negative real parts.

For a specified $x \in \Omega$, $x^\varepsilon(t)$ is a Markov diffusion process with $x^\varepsilon(0) = x$ and differential generator

$$\mathscr{L}^\varepsilon = \frac{\varepsilon}{2} \sum_{i,j} a_{i,j}^\varepsilon \frac{\partial}{\partial x_i \partial x_j} + \sum_i b_i^\varepsilon \frac{\partial}{\partial x_i} .$$

For definiteness, one can think of $a^\varepsilon, b^\varepsilon$ as being extended to all of $\mathbb{R}^d$, the process $x^\varepsilon(\cdot)$ then being obtained as below for all $t < +\infty$. We are only concerned with $x^{\varepsilon}(t)$ for $t \leq \tau_{\Omega}$, however, which does not depend on this extention. If one likes, $x^{\varepsilon}(t)$ can be considered as the solution to a stochastic differential equation (see [7] or [11])

$$dx^{\varepsilon}(t) = b^{\varepsilon}(x^{\varepsilon}(t))dt + \sqrt{\varepsilon}\, \sigma^{\varepsilon}(x^{\varepsilon}(t))d\omega_t, \; x^{\varepsilon}(0) = x$$

if $a^{\varepsilon} = \sigma^{\varepsilon}(\sigma^{\varepsilon})^T$ where σ^{ε} is Lipschitz. Alternately, $x^{\varepsilon}(t)$ can be discussed directly via the martingale problem associated with $\mathscr{L}^{\varepsilon}$; [11]. (Continuity of coefficients is sufficient for that treatment.)

The boundary functions $f^{\varepsilon}(x)$ are assumed to be bounded in x and $\varepsilon > 0$ and measurable on $\partial\Omega$. The $u^{\varepsilon}(x)$ are now defined by

$$(2.1) \qquad u^{\varepsilon}(x) = E_x^{\varepsilon}[f^{\varepsilon}(x^{\varepsilon}(\tau_{\Omega}))].$$

It can be shown that $u^{\varepsilon} \in C^2(\Omega)$ and satisfies

$$\mathscr{L}^{\varepsilon}[u^{\varepsilon}] = 0 \quad \text{in} \quad \Omega.$$

Indeed, on any ball B with $\overline{B} \subseteq \Omega$ it is true that u^{ε} is the Perron solution corresponding to the boundary data $u^{\varepsilon}|_{\partial B}$. Since Perron solutions are C^2 on the interior of their domains, for bounded measurable data, ([3], Theorem 6.11) it follows that $u^{\varepsilon} \in C^2(\Omega)$. The boundary behavior of u^{ε} does not concern us; only the boundedness in ε, x is necessary for our arguments below.

There is one more condition that we will need when proving Theorem 2 below. In deriving (4.6) we will use

$$\frac{b^{\varepsilon}(x)-b^{0}(x)}{|x|} = o(1) \quad \text{as} \quad \varepsilon \downarrow 0 \quad \text{uniformly in} \quad \Omega. \tag{2.2}$$

We know that $b^0(0) = 0$, so the above will follow from the convergence of $b^{\varepsilon}_{x_i}$ to $b^{0}_{x_i}$ if the further condition $b^{\varepsilon}(0) = 0$ is true. Once we restrict our attention to x in a compact $K \subseteq \Omega$, however, we can achieve $b^{\varepsilon}(0) = 0$ without imposing any further assumptions. The following argument accomplishes this: from the stability of 0 with respect to b^0 as in d) above and the uniform convergence of b^{ε} to b^0 one can deduce that (for sufficiently small ε) b^{ε} has a critical point ζ^{ε} such that $\zeta^{\varepsilon} \to 0$ as $\varepsilon \downarrow 0$. Change variables to $y = x - \zeta^{\varepsilon}$. The new coefficients $\tilde{a}^{\varepsilon}(y) = a^{\varepsilon}(y+\zeta^{\varepsilon})$ and $\tilde{b}^{\varepsilon}(y) = b^{\varepsilon}(y+\zeta^{\varepsilon})$ satisfy all of our assumptions above as well as $\tilde{b}^{\varepsilon}(0) = 0$. The only difficulty is that the domains $\Omega - \zeta^{\varepsilon}$ are ε-dependent. We can pass to a subdomain Ω' so that $y \in \Omega'$ implies $x = y + \zeta^{\varepsilon} \in \Omega$ and a compact subset $K' \subseteq \Omega'$ so that $x \in K$ implies $y = x - \zeta^{\varepsilon} \in K'$, for sufficiently small ε. Applying Theorem 1 to Ω', K' we get the same result as for Ω, K. Taking the details of this argument for granted, we assume in the following that $b^{\varepsilon}(0) = 0$ and consequently that (2.2) is true.

Here is our main theorem.

<u>Theorem 1</u>: Under the assumptions described above, for any compact $K \subseteq \Omega$ there exists $\delta > 0$ and $\varepsilon_0 > 0$ so that for all $0 < \varepsilon < \varepsilon_0$

$$\sup_{x,y \in K} |u^\varepsilon(x) - u^\varepsilon(y)| \le e^{-\delta/\varepsilon}.$$

Roughly, the reasoning behind the proof is that for small ε $x^\varepsilon(t)$ should, with high probability, follow the deterministic trajectory $x^0(t)$ into the vicinity of the origin before making its first excursion to the boundary $\partial\Omega$. A precise probabilistic estimate along these lines is established in the next two sections. To apply the probabilistic estimate, we need to know a modulus of continuity for u^ε. The following lemma establishes the modulus that we need; the rescaling argument is the same one used by Kamin [5].

Lemma 1: Let $K \subseteq \Omega$ be compact. Then there is a constant C so that $|\nabla u^\varepsilon(x)| \le C\varepsilon^{-1/2}$ for all $x \in K$ and $\varepsilon < 1$.

Proof:

Make the change of variables $y = \varepsilon^{-1/2}x$. Then $v^\varepsilon(y) = u^\varepsilon(\varepsilon^{1/2}y)$ satisfies

$$\frac{1}{2}\sum_{i,j} \tilde{a}^\varepsilon_{i,j}(y)V_{y_i y_j} + \sum_i \tilde{b}^\varepsilon_i(y)V_{y_i} = 0 \quad \text{for } y \in \varepsilon^{-1/2}\Omega.$$

The coefficients $\tilde{a}^\varepsilon(y) = a^\varepsilon(\varepsilon^{1/2}y)$, $\tilde{b}^\varepsilon(y) = \varepsilon^{-1/2}b^\varepsilon(\varepsilon^{1/2}y)$ are Hölder continuous with respect to y uniformly in ε. If we take r so that $B_r(x) = \{z: |x-z| < r\} \subseteq \Omega$ whenever $x \in K$, then $B_r(y) \subseteq \varepsilon^{-1/2}\Omega$ whenever $y \in \varepsilon^{-1/2}K$ and $\varepsilon < 1$. We can apply the basic Scahuder interior estimate ([3], Theorem 6.2) to $B_r(y)$ for any $y \in \varepsilon^{-1/2}K$ to conclude that $|\nabla v(y)| \leq C$ for all $y \in K$ for some constant C. This implys the lemma after changing back to the original variable x.

III: A Prototype: An Ornstein-Uhlenbeck Process

Before proving the estimate on hitting probabilities of the next section, it is convenient to look at the special case of an Ornstein-Uhlenbeck process with generator as in (3.1) below. The proof of the general case rests on a comparison with the function described by (3.4) and analyzed below.

In $\mathbb{R}^d$, $d \geq 2$, suppose that $\alpha > 0$ is a constant and $\zeta^\varepsilon(t)$ is a diffusion process with differential generator

$$\mathcal{G}^\varepsilon[u](x) = \frac{\varepsilon}{2}\,\Delta u(x) - \alpha x\cdot\nabla u(x). \tag{3.1}$$

Let $\tau(r)$ be the hitting time of the sphere of radius r:

$$\tau(r) = \inf\{t \geq 0: |\zeta^\varepsilon(t)| = r\}. \tag{3.2}$$

Take a fixed $R > 0$ and, for $r_0 < |x| < R$, define the hitting probability

$$Q^\varepsilon_{r_0}(x) = P^\varepsilon_x[\tau(r_0) < \tau(R);\ \tau(r_0) < \infty].$$

What we show is that there exists a positive constant $\delta_1 > 0$ so that

$$Q^\varepsilon_{r_0}(x) \geq 1 - e^{-\delta_1/\varepsilon} \quad \text{whenever} \quad r_0 \geq e^{-\delta_1/\varepsilon} \quad \text{and} \quad |x| < \frac{1}{3}R. \tag{3.3}$$

In words, for $|\zeta^\varepsilon(0)| < \frac{1}{3}R$ we can let $r_0 \downarrow 0$ exponentially with ε^{-1} and at the same time have the probability that

$\tau(r_0) < \tau(R)$ converging to 1 exponentially fast. To prove this we calculate $Q^{\varepsilon}_{r_0}(\cdot)$. By symmetry $Q^{\varepsilon}_{r_0}$ depends only on $r = |x|$. Thus $Q^{\varepsilon}_{r_0}(x) = Q(r)$ where $\mathscr{G}^{\varepsilon}[Q(r)] = 0$ with $Q(r_0) = 1$, $Q(R) = 0$.

$$0 = \mathscr{G}^{\varepsilon}[Q(|x|)] = \frac{\varepsilon}{2} Q''(r) + [\frac{\varepsilon}{2} \frac{d-1}{r} - \alpha r]Q'(r)$$

or

$$(3.4) \qquad Q''(r) + [\frac{\beta}{r} - \frac{2\alpha}{\varepsilon} r]Q'(r) = 0; \quad Q(r_0) = 1, \ Q(R) = 0.$$

For the above $\beta = d - 1$, but we will carry out the calculation for arbitrary positive constants α, β. Solving (3.4) gives

$$Q(r) = 1 - \frac{\int_{r_0}^{r} s^{-\beta} e^{\frac{\alpha}{\varepsilon} s^2} ds}{\int_{r_0}^{R} s^{-\beta} e^{\frac{\alpha}{\varepsilon} s^2} ds} .$$

If $r_0 < r \leq \frac{R}{3}$, then

$$\frac{\int_{r_0}^{r} s^{-\beta} e^{\frac{\alpha}{\varepsilon} s^2} ds}{\int_{r_0}^{R} s^{-\beta} e^{\frac{\alpha}{\varepsilon} s^2} ds} \leq \frac{\int_{r_0}^{\frac{1}{3}R}}{\int_{\frac{2}{3}R}^{R}} \leq \frac{\frac{1}{3} R\, e^{\frac{\alpha}{\varepsilon}(\frac{1}{3}R)^2} r_0^{-\beta}}{\frac{1}{3} R\, e^{\frac{\alpha}{\varepsilon}(\frac{2}{3}R)^2} (R)^{-\beta}} = e^{-\frac{\alpha}{\varepsilon}\frac{1}{3}R^2} r_0^{-\beta} R^{\beta}.$$

If $\log(r_0) \geq -\frac{1}{\varepsilon} \cdot (\frac{\alpha R^2}{6\beta})$, then the preceding is $\leq e^{-\frac{\alpha}{\varepsilon}\frac{1}{6}R^2} R^{\beta}$.

Consequently, if δ_1 is slightly less than the minimum of $\frac{\alpha R^2}{6}$ and $\frac{\alpha R^2}{6\beta}$ (slightly less so as to absorb the R^{β}), then for $r_0 \geq e^{-\delta_1/\varepsilon}$ and $r \leq \frac{1}{3} R$ we have

$$Q(r) \geq 1 - e^{-\delta_1/\varepsilon} \quad \text{for sufficiently small } \varepsilon > 0. \tag{3.5}$$

IV. The Hitting Probabilities in the General Case

Next, we prove that an estimate like (3.3) holds for the hitting probabilities of the process $x^{\varepsilon}(t)$. ($\tau(r)$ now denotes the time of first contact with the ball of radius r about the origin for $x^{\varepsilon}(\cdot)$.)

Theorem 2: For any compact $K \subseteq \Omega$ there exists $\delta > 0$ so that for some $\varepsilon_0 > 0$ and all $0 < \varepsilon < \varepsilon_0$

$$P_x^{\varepsilon}[\tau(r_0) < \tau_{\Omega}] \geq 1 - e^{-\delta/\varepsilon} \quad \text{whenever} \quad x \in K \quad \text{and} \quad r_0 \geq e^{-\delta/\varepsilon}.$$

Proof:

We will first make an argument for an appropriate neighborhood of the origin. The key to the proof is to use not the standard Euclidean norm $|x|$ but a different symmetric positive definite quadratic form. By hypothesis, the matrix $B = [\partial b_i^0(0)/\partial x_j]$ is stable. Lyapunov's Theorem on matricies implies that there exists a unique symmetric positive definite matrix V which solves

$$B^T V + VB = -I.$$

Define $\rho(x) = [x^T V x]^{1/2}$. For $f \in C^2(\mathbb{R})$, a computation gives that

$$(4.1)\quad \mathscr{L}^{\varepsilon}[f(\rho(x))] = f''(\rho)\cdot \frac{\varepsilon}{2} \sum_{i,j} a_{ij}^{\varepsilon}\rho_{x_i}\rho_{x_j} + f'(\rho)\cdot[\frac{\varepsilon}{2} \sum a_{ij}^{\varepsilon}\rho_{x_i x_j} + \sum b_i \rho_{x_i}],$$

$$\nabla\rho(x) = \frac{x^T V}{\rho}, \quad \rho_{x_i x_j} = \frac{V_{ij}}{\rho} + \frac{1}{\rho^3} \sum_{\ell,k} V_{i\ell} x_\ell x_k V_{kj}$$

The idea is to effect a comparison of each of the terms of $\mathcal{L}^{\varepsilon}[f(\rho)]$ with those of $\mathcal{G}^{\varepsilon}[Q(r)]$ in (3.4). First,

$$\sum a_{ij}^{\varepsilon}\rho_{x_i}\rho_{x_j} = \frac{x^T V a^{\varepsilon} V x}{x^T V x}$$

which is bounded above and below away from 0 on $\Omega - \{0\}$. Moreover, these bounds can be taken to be independent of ε sufficiently small since $a^{\varepsilon} \to a^0$ uniformly. Thus, there exists a constant $A > 0$ so that

$$(4.2) \qquad A^{-1} \le \sum a_{ij}^{\varepsilon}\rho_{x_i}\rho_{x_j} \le A \quad \text{in} \quad \Omega - \{0\}, \text{ all sufficiently small } \varepsilon.$$

Secondly,

$$(4.3) \qquad \sum_{i,j} a_{ij}^{\varepsilon}\rho_{x_i x_j} = \frac{1}{\rho}\sum a_{ij}^{\varepsilon}V_{ij} + \frac{1}{\rho}\,\frac{x^T V a^{\varepsilon} V x}{\rho^2} \le \frac{C}{\rho}$$

for a positive constant C (again uniformly in ε sufficiently small). Thirdly,

$$\sum b_i^{\varepsilon}\rho_{x_i} = \frac{x^T V b\ (x)}{\rho(x)} = \frac{x^T V b^0(x)}{\rho(x)} + \frac{x^T V(b^{\varepsilon}-b^0)}{\rho}$$

Using $b^0(x) = Bx + o(|x|)$ and $b^{\varepsilon}(x) - b^0(x) = |x|\,o(1)$ from (2.2), we find that

$$\begin{aligned}\sum b_i^{\varepsilon}\rho_{x_i} &= \rho\cdot\left[\frac{x^T V B x}{\rho^2} + o(|x|) + o(1)\right] \\ &= \rho\cdot\left[-\frac{1}{2}\,\frac{|x|^2}{\rho^2} + o(|x|) + o(1)\right].\end{aligned}$$

(The $o(|x|)$ is an $|x| \to 0$ and is independent of ε. The $o(1)$ is as $\varepsilon \to 0$ and is uniform in x.) The second equality is a consequence of our choice of V. It follows that for some D,R and ε_0 all positive,

$$(4.4) \qquad \sum b_i^\varepsilon \rho_{x_i} \le -D\rho(x) \quad \text{if} \quad \rho(x) \le R \quad \text{and} \quad \varepsilon < \varepsilon_0.$$

(Also restrict R so that $x \in \Omega$ whenever $\rho(x) < R$.) Take $\alpha = DA^{-1}$, $\beta = AC$ and then $Q(\cdot)$ as in (3.2). Since $Q' \le 0$, (4.1)-(4.4) combined imply that, for $\varepsilon < \varepsilon_0$ and $\rho(x) \le R$.

$$(4.5) \qquad \mathscr{L}^\varepsilon[Q(\rho(x))] \ge \frac{\varepsilon}{2} A^{-1} \cdot \{Q''(\rho) + Q'(\rho)[\frac{AC}{\rho} - \frac{2}{\varepsilon} AD\rho]\} = 0.$$

Using $\tilde{\tau}(r)$ for the hitting time of the set $\{x: \rho(x) = r\}$ by $x^\varepsilon(\cdot)$, (4.5) implies, either via the maximum principle or the fact that $Q(\rho(x^\varepsilon(t)))$ is a submartingale, that

$$P_x^\varepsilon[\tilde{\tau}(r_0) < \tilde{\tau}(R)] \ge Q(\rho(x)).$$

If $\gamma > 0$ is an constant so that $\gamma \le \frac{\rho(x)}{|x|} \le \gamma^{-1}$, then $\tilde{\tau}(\gamma r_0) > \tau(r_0)$ provided $|x^\varepsilon(0)| > r_0$. If $|x^\varepsilon(0)| < \gamma R$, then $\rho(x^\varepsilon(0)) < R$ and $\tilde{\tau}(R) \le \tau_\Omega$. Consequently, for $r_0 < |x^\varepsilon(0)| < \gamma R$ we have

$$P_x^\varepsilon[\tau(r_0) < \tau_\Omega] \ge P_x^\varepsilon[\tilde{\tau}(\gamma r_0) < \tilde{\tau}(R)] \ge Q(\rho(x)).$$

This is trivially true also if $|x^\varepsilon(0)| \le r_0$. The calculation of Section 2 now implies the existence of $\delta_1 > 0$ so that for all $0 < \varepsilon < \varepsilon_0$ and $|x| \le \frac{\gamma}{3} R$

$$(4.6) \qquad P_x^{\varepsilon}[\tau(r_0) < \tau_{\Omega}] \geq 1 - e^{-\delta_1/\varepsilon} \quad \text{if} \quad \gamma r_0 \geq e^{-\delta_1/\varepsilon}.$$

The last step of the proof is to show that such an estimate remains true for all $x \in K$. By the strong Markov property

$$P_x^{\varepsilon}[\tau(r_0) < \tau_{\Omega}] = E_x^{\varepsilon}[P^{\varepsilon}_{x^{\varepsilon}(\tau(\frac{\gamma}{3} R))}[\tau(r_0) < \tau_{\Omega}];\ \tau(\tfrac{\gamma}{3} R) < \tau_{\Omega}]$$

$$\geq (1-e^{-\delta_1/\varepsilon}) \cdot P_x^{\varepsilon}[\tau(\tfrac{\gamma}{3} R) < \tau_{\Omega}].$$

It is sufficient therefore to prove that for some $\delta_2 > 0$ and all $x \in K$,

$$(4.7) \qquad P_x^{\varepsilon}[\tau(\tfrac{\gamma}{3} R) < \tau_{\Omega}] \geq 1 - e^{-\delta_2/\varepsilon}.$$

Let $\phi^{\varepsilon}(t;x)$, $\varepsilon \geq 0$, denote the solution of the deterministic equation $\phi'(t) = b^{\varepsilon}(\phi(t))$ with $\phi(0) = x$. Since K is contained in the domain of attraction of the stable point 0, there exist $T, \eta > 0$ so that if $x \in K$ then

$y \in \Omega$ whenever $|y - \varphi^0(s;x)| < 2\eta$ for some $0 \leq s \leq T$,

and

$|y| < \frac{\gamma}{3} R$ whenever $|y - \phi^0(T;x)| < 2\eta$.

As $\varepsilon \to 0$, b^{ε} converges to b^0 uniformly in Ω and consequently $\phi^{\varepsilon}(t;x)$ converges to $\phi^0(t;x)$ uniformly for $t \in [0,T]$ and $x \in K$. Therefore, if ε is sufficiently small and $x \in K$ it will be true that

$$y \in \Omega \quad \text{whenever} \quad |y - \phi^{\varepsilon}(s;x)| < \eta \quad \text{for some} \quad 0 < s < T,$$

and

$$|y| < \frac{\gamma}{3} R \quad \text{whenever} \quad |y - \phi^{\varepsilon}(T;x)| < \eta.$$

For such ε and $x \in K$,

$$P_x^{\varepsilon}[\tau_{\Omega} \le \tau(\frac{\gamma}{3} R)] \le P_x^{\varepsilon}[\sup_{0 \le s \le T} |\phi^{\varepsilon}(s,x) - x^{\varepsilon}(s)| \ge \eta].$$

Define

$$\theta^{\varepsilon}(t) = x^{\varepsilon}(t) - x - \int_0^t b^{\varepsilon}(x^{\varepsilon}(s))ds$$

$\left(= \sqrt{\varepsilon} \int_0^t \sigma^{\varepsilon}(x^{\varepsilon}(s))ds \text{ if } x^{\varepsilon} \text{ is obtained from an Itô equation}\right)$.

Gronwall's inequality implies that

$$\sup_{[0,T]} |\phi^{\varepsilon}(s;x) - x^{\varepsilon}(s)| \le e^{MT} \cdot \sup_{0 \le s \le T} |\theta^{\varepsilon}(s)|$$

where M is the Lipschitz constant for the $b^{\varepsilon}(\cdot)$ (uniform in ε). It is a standard argument, using exponential martingales, that

$$P_x^{\varepsilon}[\sup_{[0,T]} |\theta^{\varepsilon}(s)| \ge \ell] \le (2d)\exp[\frac{-\ell^2}{2d\varepsilon AT}]$$

where $x^T a^{\varepsilon} x \le A||x||^2$; see [11], equation (2.1), pg. 87 and proof for instance. Combining these facts, for all $x \in K$ and ε sufficiently small,

$$P_x^{\varepsilon}[\tau_\Omega \le \tau(\tfrac{\gamma}{3} R)] \le (2d)\exp[-\tfrac{1}{\varepsilon}(\tfrac{\eta^2 e^{-2MT}}{2dAT})].$$

This shows (4.7) and completes the proof.

Theorem 1 is now simple.

Proof (of Theorem 1):

Take any $x \in K$ and set $r_0 = e^{-\delta/\varepsilon}$ ($\delta > 0$ as in Theorem 2).

$$u^{\varepsilon}(x) = E_x^{\varepsilon}[u^{\varepsilon}(x^{\varepsilon}(\tau_\Omega(r_0))); \tau(r_0) < \tau_\Omega] + E_x^{\varepsilon}[f^{\varepsilon}(x^{\varepsilon}(\tau_\Omega)); \tau_\Omega \le \tau(r_0)].$$

Therefore,

$$\begin{aligned} u^{\varepsilon}(x) - u^{\varepsilon}(0) &= E_x^{\varepsilon}[u^{\varepsilon}(x^{\varepsilon}(\tau_\Omega(r_0))) - u^{\varepsilon}(0); \tau(r_0) < \tau_\Omega] \\ &\quad + E_x^{\varepsilon}[f^{\varepsilon}(x^{\varepsilon}(\tau_\Omega)) - u^{\varepsilon}(0); \tau_\Omega \le \tau(r_0)] \\ |u^{\varepsilon}(x) - u^{\varepsilon}(0)| &\le \sup_{|y|\le r_0} |u^{\varepsilon}(y) - u^{\varepsilon}(0)| + \\ &\quad 2 \sup_{\partial\Omega}|f^{\varepsilon}|\cdot P_x^{\varepsilon}[\tau_\Omega \le \tau(r_0)] \\ &\le \sup_K |\nabla u^{\varepsilon}|\cdot r_0 + 2 \sup_{\partial\Omega}|f^{\varepsilon}| e^{-\delta/\varepsilon}. \\ &\le C\varepsilon^{-1/2} e^{-\delta/\varepsilon} + 2 \sup_{\partial\Omega}|f^{\varepsilon}| e^{-\delta/\varepsilon}. \end{aligned}$$

The theorem now follows (with a new slightly smaller δ).

V: Concluding Remarks

We have two simple observations to make in closing. The first is regarding the case in which Ω contains several critical points of (1.1). This has been discussed in the literature: [9], [12]. Both $u^\varepsilon \to$ constant and $u^\varepsilon \to$ a piecewise constant function are possibilities now, depending on the Ventcel-Freidlin variational distances between the critical points and $\partial\Omega$. If x^* is an asymptotically stable critical point (replacing the origin in d) of section 2) and $\Omega^* \subseteq \Omega$ is its domain of attraction, then by taking $f^\varepsilon = u^\varepsilon$ on $\partial\Omega^*$ we can apply Theorem 1 to see that leveling takes place exponentially fast in each such domain of attraction.

Finally, we observe that the specification of boundary data f^ε is actually superfluous. All that matters in the proof is the availability of a bound in ε for the u^ε. Theorem 1 could be formulated as follows:

for $K \subseteq \Omega$ compact there exist $\delta > 0$ and $\varepsilon_0 > 0$ so that whenever $\mathscr{L}^\varepsilon[u] = 0$ in Ω and $\varepsilon < \varepsilon_0$,

$$\sup_{x,y \in K} |u(x) - u(y)| \le e^{-\delta/\varepsilon} \sup_\Omega |u|. \tag{6.1}$$

Define the exit measures on the Borel subsets of $\partial\Omega$ by

$$\pi_x^\varepsilon(B) = P_x^\varepsilon[x^\varepsilon(\tau_\Omega) \in B].$$

The strong maximum principle implies that π_x^ε and π_y^ε are mutually absolutely continuous for $x,y \in \Omega$. (6.1) implies that

$$\left|\int_{\partial\Omega} f(s)\left(1 - \frac{d\pi_y^\varepsilon}{d\pi_x^\varepsilon}\right)\pi_x^\varepsilon(ds)\right| \leq e^{-\delta/\varepsilon}\|f\|_{L^\infty(\pi_x^\varepsilon)}$$

for all f bounded and measurable on $\partial\Omega$. This is equivalent to

$$\left\|1 - \frac{d\pi_y^\varepsilon}{d\pi_x^\varepsilon}\right\|_{L^1(\pi_x^\varepsilon)} \leq e^{-\delta/\varepsilon} \quad \text{for } x,y \in K,\ \varepsilon < \varepsilon_0. \tag{6.2}$$

In cases for which a Green's function exists (if $\partial\Omega$ and all the coefficients are C^2 for instance) so that u^ε can be expressed as

$$u^\varepsilon(x) = \int_{\partial\Omega} G^\varepsilon(x,s)f^\varepsilon(s)ds,$$

then $\pi_x^\varepsilon(ds) = G^\varepsilon(x,s)ds$ on $\partial\Omega$, and (6.2) becomes, for $x,y \in K$,

$$\int_{\partial\Omega}|G^\varepsilon(x,s) - G^\varepsilon(y,s)|ds \leq e^{-\delta/\varepsilon}. \tag{6.3}$$

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